

# Profile decomposition for sequences of Borel measures

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*To the memory of Mircea Reghiș*

## Abstract

We prove that, if dichotomy occurs when the concentration-compactness principle is used, the dichotomizing sequence can be chosen so that a nontrivial part of it concentrates. Iterating this argument leads to a profile decomposition for arbitrary sequences of bounded Borel measures. To illustrate our results we give an application to the structure of bounded sequences in the Sobolev space  $W^{1,p}(\mathbf{R}^N)$ .

**Keywords.** Concentration-compactness principle, profile decomposition, Sobolev spaces.

## 1 Introduction

The concentration-compactness principle, introduced by P.-L. Lions in his celebrated papers [13], [14] is an extremely effective tool in many areas in Mathematics and has a huge number of applications. In its simplest form the principle asserts that given a sequence  $(\mu_n)_{n \geq 1}$  of probability measures on  $\mathbf{R}^N$ , there exists a subsequence  $(\mu_{n_k})_{k \geq 1}$  that either spreads over the space so that the measure of each ball tends to zero uniformly with respect to the size of the ball (this phenomenon is called "vanishing"; see Definition 1.2 below for a precise statement), or  $\mu_{n_k}$  can be "split" into two nontrivial parts whose supports are far away from each other (this situation is called "dichotomy"), or most of the mass of  $\mu_{n_k}$  remains on balls of fixed radius (the latter case is called "concentration"; for a precise statement see Definition 1.1).

Many variants of the concentration-compactness principle exist in the literature (see, e.g., [15], [16]; for "abstract" versions we refer to [22], [24]). An important refinement is the so-called "profile decomposition" of sequences of functions. Early variants of profile decomposition for sequences of approximate solutions of some PDE can already be found in [2] and [23]. For arbitrary bounded sequences in Sobolev spaces, profile decomposition was fully developed by P. Gérard ([1], [5], [6]). In the context of Strichartz estimates for the Schrödinger group  $e^{it\Delta}$ , the first profile decomposition result was obtained by F. Merle and L. Vega [19]. This technique has led to important applications in the theory of nonlinear dispersive equations (see e.g. [3], [4], [7], [8], [10], [11], [12], [19], [21] and references therein).

In most applications of the concentration-compactness principle, the difficult part is to understand the dichotomy case. We show in Lemma 1.3 that whenever dichotomy occurs, we can choose a "dichotomizing subsequence"  $\mu_{n_k} = \mu_{k,1} + \mu_{k,2} + o(1)$  such that  $\mu_{k,1}$  and  $\mu_{k,2}$  have supports far away from each other and, in addition, the sequence  $(\mu_{k,1})_{k \geq 1}$  "concentrates." Iterating this argument we are able to prove a profile decomposition result for arbitrary sequences of bounded Borel measures. The spectrum of possible applications seems large, starting from the existence of minimizers in Calculus of Variations to possible extensions of the blowup theory for nonlinear dispersive equations initiated by C. Kenig and F. Merle ([10], [11]). For instance, Theorem 1.5 below has already been crucial in proving the main results in [18].

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**Definitions and notation.** For the convenience of the reader we collect here the basic notions used throughout the paper.

**Definition 1.1** Let  $(\mathcal{X}, d)$  be a metric space and let  $(\mu_n)_{n \geq 1}$  be a sequence of positive Borel measures on  $\mathcal{X}$ . Let  $(x_n)_{n \geq 1}$  be a sequence of points in  $\mathcal{X}$ . We say that  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$  if for any  $\varepsilon > 0$  there exists  $R_\varepsilon > 0$  such that  $\mu_n(\mathcal{X} \setminus B(x_n, R_\varepsilon)) < \varepsilon$  for all  $n \geq 1$ .

Sometimes we simply say that  $(\mu_n)_{n \geq 1}$  concentrates if there is a sequence of points around which  $(\mu_n)_{n \geq 1}$  concentrates. If  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$  it is obvious that any subsequence  $(\mu_{n_k})_{k \geq 1}$  concentrates (around the corresponding subsequence  $(x_{n_k})_{k \geq 1}$ ).

**Definition 1.2** Let  $(\mathcal{X}, d)$  be a metric space and  $(\mu_n)_{n \geq 1}$  a sequence of positive Borel measures on  $\mathcal{X}$ . We say that  $(\mu_n)_{n \geq 1}$  is a vanishing sequence if for any  $r > 0$ ,

$$\sup_{x \in \mathcal{X}} \mu_n(B(x, r)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Given a positive Borel measure  $\mu$  on a metric space  $\mathcal{X}$ , the concentration function of  $\mu$  is  $q : [0, \infty) \rightarrow [0, \infty)$  defined by  $q(t) = \sup_{x \in \mathcal{X}} \mu(B(x, t))$ . It is clear that  $q$  is nondecreasing and  $\lim_{t \rightarrow \infty} q(t) = \mu(\mathcal{X})$ . If  $(\mu_n)_{n \geq 1}$  is as above and  $q_n$  is the concentration function of  $\mu_n$ , it is obvious that  $(\mu_n)_{n \geq 1}$  is a vanishing sequence if and only if  $\limsup_{n \rightarrow \infty} q_n(t) = 0$  for all  $t > 0$ , or

$$\text{equivalently } \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n(t) \right) = 0.$$

Assume that  $\mu$  is a Borel measure on  $\mathcal{X}$ . If  $M \subset \mathcal{X}$  is a Borel set, we denote by  $\mu|_M$  the restriction of  $\mu$  to  $M$  defined by  $\mu|_M(E) = \mu(M \cap E)$ . Let  $(\rho_n)_{n \geq 1} \subset L^1(\mathcal{X}, \mu)$ . Each  $\rho_n$  generates a Borel measure  $\mu_n$  defined by  $\mu_n(E) = \int_E \rho_n d\mu$ . The concentration function of  $\rho_n$  is the same as that of  $\mu_n$ . We say that  $(\rho_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$ , respectively that it is a vanishing sequence, if  $(\mu_n)_{n \geq 1}$  has the same property.

**Main results.** Our main result is Theorem 1.5 below. Its proof relies on the next lemma, which is of independent interest.

**Lemma 1.3** Let  $(\mathcal{X}, d)$  be a metric space and let  $(\mu_n)_{n \geq 1}$  be a sequence of positive Borel measures on  $\mathcal{X}$  such that  $\mu_n(\mathcal{X}) \leq M$  for all  $n$ , where  $M$  is a positive constant. Denote by  $q_n(t) = \sup_{x \in \mathcal{X}} \mu_n(B(x, t))$  the concentration function of  $\mu_n$ . Assume that

$$(1.1) \quad q_n(t) \longrightarrow q(t) \quad \text{as } n \longrightarrow \infty \text{ for a.e. } t > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} q(t) = \alpha > 0.$$

Fix an increasing sequence  $t_n \rightarrow \infty$  such that  $q_n(t_n) \rightarrow \alpha$  (the existence of such sequence is guaranteed by Lemma 2.2 (ii)-(iii) below).

There exist a subsequence  $(\mu_{n_k})_{k \geq 1}$ , a sequence of points  $(x_k)_{k \geq 1} \subset \mathcal{X}$  and a sequence  $(R_k)_{k \geq 1}$  such that  $R_k \leq t_{n_k}$ ,  $R_k \rightarrow \infty$ ,  $\mu_{n_k}(B(x_k, R_k)) \rightarrow \alpha$  as  $k \rightarrow \infty$ , and the sequence of measures  $(\mu_{n_k}|_{B(x_k, R_k)})_{k \geq 1}$  concentrates around  $(x_k)_{k \geq 1}$ .

**Remark 1.4** If  $(\mu_{n_k}|_{B(x_k, R_k)})_{k \geq 1}$  concentrates around  $(x_k)_{k \geq 1}$ , it follows from Remark 2.1 (i) below that for any sequence  $t_k \rightarrow \infty$ ,  $t_k < R_k$ , the sequence  $(\mu_{n_k}|_{B(x_k, t_k)})_{k \geq 1}$  also concentrates around  $(x_k)_{k \geq 1}$  and  $\mu_n(B(x_k, R_k) \setminus B(x_k, t_k)) \rightarrow 0$  as  $k \rightarrow \infty$ . In particular, for any  $\varphi : [0, \infty) \rightarrow [0, \infty)$  such that  $\varphi(t) < t$  and  $\varphi(t) \rightarrow \infty$  as  $t \rightarrow \infty$  we have  $\mu_{n_k}(B(x_k, R_k) \setminus B(x_k, \varphi(R_k))) \rightarrow 0$  as  $k \rightarrow \infty$ . In applications it is very important to dispose of a "large" region  $B(x_k, R_k) \setminus B(x_k, \varphi(R_k))$  with "small" measure in order to perform a convenient cut-off (see also Remark 1.6 below).

**Theorem 1.5** Let  $(\mathcal{X}, d)$  be a metric space and  $(\mu_n)_{n \geq 1}$  a sequence of positive Borel measures on  $\mathcal{X}$  such that

$$M := \limsup_{n \rightarrow \infty} \mu_n(\mathcal{X}) < \infty.$$

Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be an increasing function such that  $\varphi(s) \leq \frac{s}{2}$  for all  $s$  and  $\lim_{s \rightarrow \infty} \varphi(s) = \infty$ .

Then either  $(\mu_n)_{n \geq 1}$  is a vanishing sequence, or there exists an increasing mapping  $j : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that the subsequence  $(\mu_{j(n)})_{n \geq 1}$  satisfies one of the following properties:

i) There are  $k \in \mathbf{N}^*$ , positive numbers  $m_1, \dots, m_k$ , sequences of points  $(x_n^i)_{n \geq 1} \subset \mathcal{X}$  and increasing sequences of positive numbers  $(r_n^i)_{n \geq 1}$  such that  $r_n^i \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $i \in \{1, \dots, k\}$  satisfying the following properties:

- a) For each  $n$  the balls  $B(x_n^i, r_n^i)$ ,  $i \in \{1, \dots, k\}$  are disjoint.
- b) For each  $i \in \{1, \dots, k\}$  we have

$$\mu_{j(n)}(B(x_n^i, \varphi(r_n^i))) \rightarrow m_i \quad \text{as } n \rightarrow \infty, \quad \mu_{j(n)}(B(x_n^i, r_n^i) \setminus B(x_n^i, \varphi(r_n^i))) \leq \frac{1}{2^{n+i}}.$$

and the sequence of measures  $(\mu_{j(n)}|_{B(x_n^i, r_n^i)})_{n \geq 1}$  concentrates around  $(x_n^i)_{n \geq 1}$ .

c) The sequence of measures  $(\mu_{j(n)}|_{\mathcal{X} \setminus \cup_{i=1}^k B(x_n^i, r_n^i)})_{n \geq 1}$  is a vanishing sequence.

ii) There are positive numbers  $m_1, \dots, m_k, \dots$  such that  $m_{k+1} \leq 2m_k$ , sequences of points  $(x_n^i)_{n \geq i} \subset \mathcal{X}$  and increasing sequences of positive numbers  $(r_n^i)_{n \geq i}$  such that  $r_n^k \rightarrow \infty$  as  $n \rightarrow \infty$  for each fixed  $k$  and the following properties hold:

- a) For each  $n$  the balls  $B(x_n^1, r_n^1), \dots, B(x_n^n, r_n^n)$  are disjoint.
- b) The same as (b) in (i) above.
- c) Denote by  $\tilde{q}_n^\ell$  is the concentration function of  $\mu_{j(n)}|_{\mathcal{X} \setminus \cup_{i=1}^\ell B(x_n^i, r_n^i)}$  for  $\ell \geq n$ . Then

$$\lim_{\ell \rightarrow \infty} \left( \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} \tilde{q}_n^\ell(t) \right) \right) = 0.$$

d) The sequence of measures  $(\mu_{j(n)}|_{\mathcal{X} \setminus \cup_{i=1}^n B(x_n^i, r_n^i)})_{n \geq 1}$  is a vanishing sequence.

**Remark 1.6** Let  $(\varphi_k)_{k \geq 1}$  be a sequence of increasing functions, where each  $\varphi_k$  is as in Theorem 1.5 and  $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_k \geq \varphi_{k+1} \geq \dots$ . Theorem 1.5 above still holds true (with obvious modifications in the proof) if (ii) (b) is replaced by

$$\mu_{j(n)}(B(x_n^k, \varphi_k(r_n^k))) \rightarrow m_k \quad \text{as } n \rightarrow \infty, \quad \mu_{j(n)}(B(x_n^k, r_n^k) \setminus B(x_n^k, \varphi_k(r_n^k))) \leq \frac{1}{2^{n+k}}.$$

The region  $B(x_n^k, r_n^k) \setminus B(x_n^k, \varphi_k(r_n^k))$  has small measure and, in applications, is often used to perform an appropriate "cut-off." The fact that the  $\varphi_k$ 's may depend on  $k$  and may tend to infinity as slowly as we want gives even more freedom to perform the cut-off. We refer to [18] for an application ( $\varphi(t) = t^{\frac{1}{3}}$  has been used there).

As an application we prove a "profile decomposition" result for bounded sequences in the Sobolev space  $W^{1,p}(\mathbf{R}^N)$ . Our proof is simple, direct and relies only on Theorem 1.5. Similar results already exist in the literature. To our knowledge, one of the first results of this kind was

Theorem 1.1 p. 214 in [5] which deals with bounded sequences in  $\dot{H}^s(\mathbf{R}^N)$ ,  $0 < s < \frac{N}{2}$ . The proofs in [5] use in an essential way the Fourier transform and improved Sobolev inequalities. An extension to  $\dot{H}^{s,p}(\mathbf{R}^N)$  spaces is given by Theorem 1 p. 386 in [9]. The proof in [9] relies on wavelet decomposition and improved Sobolev inequalities. We have to mention the recent work [20], which recovers the profile decomposition in [5] in  $\dot{H}^s$  (among other interesting results) and uses again improved Sobolev estimates. The following result is a generalization to  $W^{1,p}$  spaces of Proposition 3.1 p. 2822 in [7] and of Proposition 2.1 p. 1039 in [8], which deal with bounded sequences in  $H^1(\mathbf{R}^N)$ . Unlike in the previous works we do not use the Fourier transform and improved Sobolev inequalities, nor the Hilbert space structure. We postpone to a subsequent work a similar study of bounded sequences in the spaces  $W^{s,p}(\mathbf{R}^N)$  as well as in the spaces  $\dot{W}^{1,p}(\mathbf{R}^N)$  (the latter case is a little bit more delicate because of scaling invariance, but Theorem 1.5 can still be used to get a profile decomposition result in the spirit of [5], [1]).

**Theorem 1.7** *Let  $1 < p < \infty$  and let  $(u_n)_{n \geq 1}$  be a bounded sequence in  $W^{1,p}(\mathbf{R}^N)$ . Let  $p^* = \frac{Np}{N-p}$  if  $p < N$  and  $p^* = \infty$  if  $p \geq N$ . There exists a subsequence of  $(u_n)_{n \geq 1}$  (still denoted the same), a family  $(V^i)_{i \in \mathbf{N}^*}$  of functions in  $W^{1,p}(\mathbf{R}^N)$  and a family of sequences  $(y_n^i)_{n \geq 1} \subset \mathbf{R}^N$ ,  $i \in \mathbf{N}^*$ , such that:*

i) *For any  $i \neq j$  we have  $|y_n^i - y_n^j| \rightarrow \infty$  as  $n \rightarrow \infty$ .*

ii) *For any  $k \in \mathbf{N}^*$  there holds*

$$u_n = \sum_{i=1}^k V^i(\cdot - y_n^i) + w_n^k, \quad \text{where } \lim_{k \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} \|w_n^k\|_{L^q(\mathbf{R}^N)} \right) = 0 \text{ for all } q \in (p, p^*).$$

iii) *For all  $k \geq 1$  we have  $\|u_n\|_{L^p(\mathbf{R}^N)}^p = \sum_{i=1}^k \|V^i\|_{L^p(\mathbf{R}^N)}^p + \|w_n^k\|_{L^p(\mathbf{R}^N)}^p + o(1)$  as  $n \rightarrow \infty$ .*

iv) *If  $p = 2$  we have  $\|\nabla u_n\|_{L^2(\mathbf{R}^N)}^2 = \sum_{i=1}^k \|\nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + \|\nabla w_n^k\|_{L^2(\mathbf{R}^N)}^2 + o(1)$  as  $n \rightarrow \infty$ .*

As pointed out in [9] p. 387, the equality in Theorem 1.7 (iv) may be false if  $p \neq 2$ .

The rest of this paper is organized as follows. In the next section we give some elementary results that will be used in proofs. We prove Lemma 1.3 in section 3 and Theorem 1.5 in section 4. The proof of Theorem 1.7 is given in section 5.

## 2 Some elementary facts

In this section we collect some simple observations that will be very useful in the sequel.

**Remark 2.1** Let  $(\mu_n)_{n \geq 1}$  be a sequence of positive Borel measures on  $\mathcal{X}$ .

(i) Assume that  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$ . If  $(M_n)_{n \geq 1}$  are any Borel measurable sets, the sequence  $(\mu_n|_{M_n})_{n \geq 1}$  also concentrates around  $(x_n)_{n \geq 1}$ . If  $(t_n)_{n \geq 1}$  is any sequence of positive numbers such that  $t_n \rightarrow \infty$ , then  $\mu_n(\mathcal{X} \setminus B(x_n, t_n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

(ii) If  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$  and  $(y_n)_{n \geq 1}$  is any sequence of points such that  $(d(x_n, y_n))_{n \geq 1}$  is bounded, then  $(\mu_n)_{n \geq 1}$  concentrates around  $(y_n)_{n \geq 1}$ .

(iii) Denote  $\alpha_n = \mu_n(\mathcal{X})$ . Assume that each  $\alpha_n$  is finite and  $\alpha_n \geq \alpha_0$ , where  $\alpha_0$  is a positive constant. Let  $q_n(r) = \sup_{x \in \mathcal{X}} \mu_n(B(x, r))$  be the concentration function of  $\mu_n$ . Then  $(\mu_n)_{n \geq 1}$  concentrates if and only if

$$(2.1) \quad \forall \varepsilon > 0, \exists r_\varepsilon > 0, \forall r \geq r_\varepsilon, \forall n \geq 1, \quad 0 \leq \alpha_n - q_n(r) < \varepsilon.$$

*Proof.* If  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$  and  $R_\varepsilon$  is as in Definition 1.1, for any  $r \geq R_\varepsilon$  we have  $\alpha_n \geq q_n(r) \geq q_n(R_\varepsilon) \geq \mu_n(B(x_n, R_\varepsilon)) > \alpha_n - \varepsilon$  and (2.1) holds.

Conversely, assume that (2.1) is satisfied and fix  $\varepsilon \in (0, \frac{\alpha_0}{2})$ . Let  $r_\varepsilon$  be given by (2.1). For all  $n$  we have  $q_n(r_\varepsilon) > \alpha_n - \varepsilon$ , hence we may choose  $x_n \in \mathcal{X}$  such that  $\mu_n(B(x_n, r_\varepsilon)) > \alpha_n - \varepsilon$ . Let  $0 < \varepsilon' < \varepsilon$ . Choose  $r_{\varepsilon'}$  such that (2.1) holds. In particular we have  $q_n(r_{\varepsilon'}) > \alpha_n - \varepsilon'$  for all  $n$ , hence there exist  $y_{n, \varepsilon'} \in \mathcal{X}$  such that  $\mu_n(B(y_{n, \varepsilon'}, r_{\varepsilon'})) > \alpha_n - \varepsilon'$ . The balls  $B(x_n, \varepsilon)$  and  $B(y_{n, \varepsilon'}, r_{\varepsilon'})$  must intersect (for otherwise we would have  $\alpha_n = \mu_n(\mathcal{X}) \geq \mu_n(B(x_n, \varepsilon)) + \mu_n(B(y_{n, \varepsilon'}, r_{\varepsilon'})) > \alpha_n - \varepsilon + \alpha_n - \varepsilon' > \alpha_n$ , a contradiction). Hence  $B(y_{n, \varepsilon'}, r_{\varepsilon'}) \subset B(x_n, r_\varepsilon + 2r_{\varepsilon'})$ . Taking  $R_{\varepsilon'} = r_\varepsilon + 2r_{\varepsilon'}$  for  $0 < \varepsilon' < \varepsilon$  we have  $\mu_n(\mathcal{X} \setminus B(x_n, R_{\varepsilon'})) \leq \mu_n(\mathcal{X} \setminus B(y_{n, \varepsilon'}, r_{\varepsilon'})) < \varepsilon'$ , therefore  $(\mu_n)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$ .

(iv) We keep the previous notation and we assume that  $(\alpha_n)_{n \geq 1}$  is bounded,  $0 < \alpha_0 \leq \alpha_n$  for all  $n \geq 1$  and there exists an increasing sequence  $t_k \rightarrow \infty$  such that for any  $k$  the sequence  $(q_n(t_k))_{n \geq 1}$  converges. Denote  $q(t_k) = \lim_{n \rightarrow \infty} q_n(t_k)$  and  $Q = \lim_{k \rightarrow \infty} q(t_k)$  ( $Q$  exists because  $(q(t_k))_{k \geq 1}$  is nondecreasing). Then we have  $Q \leq \liminf_{n \rightarrow \infty} \alpha_n$ . Moreover,  $(\mu_n)_{n \geq 1}$  concentrates if and only if  $\lim_{n \rightarrow \infty} \alpha_n = Q$ .

*Proof.* We have  $q_n(t_k) \leq \alpha_n$  for all  $n$  and  $k$ , hence  $q(t_k) \leq \liminf_{n \rightarrow \infty} \alpha_n$  for all  $k$  and the first claim follows.

Assume that  $(\mu_n)_{n \geq 1}$  concentrates. Fix  $\varepsilon > 0$ . Let  $r_\varepsilon$  be as in (2.1). There is  $k_\varepsilon$  such that  $t_{k_\varepsilon} > r_\varepsilon$  and  $|q(t_{k_\varepsilon}) - Q| < \varepsilon$ . By (2.1) we have  $\alpha_n - \varepsilon < q_n(t_k) \leq \alpha_n$  for all  $k \geq k_\varepsilon$  and  $n \geq 1$ . Choose  $n_\varepsilon$  such that  $|q_n(t_{k_\varepsilon}) - q(t_{k_\varepsilon})| < \varepsilon$  for  $n \geq n_\varepsilon$ . Then

$$|\alpha_n - Q| \leq |\alpha_n - q_n(t_{k_\varepsilon})| + |q_n(t_{k_\varepsilon}) - q(t_{k_\varepsilon})| + |q(t_{k_\varepsilon}) - Q| < 3\varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

Since  $\varepsilon$  is arbitrary we infer that  $\alpha_n \rightarrow Q$  as  $n \rightarrow \infty$ .

Conversely, assume that  $\alpha_n \rightarrow Q$ . Fix  $\varepsilon > 0$ . Choose  $k_\varepsilon$  such that  $|q(t_{k_\varepsilon}) - Q| < \frac{\varepsilon}{3}$ , then choose  $n_\varepsilon$  such that  $|\alpha_n - Q| < \frac{\varepsilon}{3}$  and  $|q_n(t_{k_\varepsilon}) - q(t_{k_\varepsilon})| < \frac{\varepsilon}{3}$  for all  $n \geq n_\varepsilon$ . Since  $q_n$  is nondecreasing, for all  $t \geq t_{k_\varepsilon}$  and  $n \geq n_\varepsilon$  we have

$$0 \leq \alpha_n - q_n(t) \leq \alpha_n - q_n(t_{k_\varepsilon}) \leq |\alpha_n - Q| + |Q - q(t_{k_\varepsilon})| + |q(t_{k_\varepsilon}) - q_n(t_{k_\varepsilon})| < \varepsilon.$$

Since  $q_n(t) \rightarrow \alpha_n$  as  $t \rightarrow \infty$  for any fixed  $n$ , we may choose  $r_\varepsilon \geq t_{k_\varepsilon}$  such that  $0 \leq \alpha_n - q_n(t) < \varepsilon$  for all  $n \in \{1, \dots, n_\varepsilon - 1\}$  and  $t \geq r_\varepsilon$ . Hence (2.1) holds, that is  $(\mu_n)_{n \geq 1}$  concentrates.

We will repeatedly use the following lemmas.

**Lemma 2.2** *Let  $(f_n)_{n \geq 1}$  be a sequence of nondecreasing functions,  $f_n : [0, \infty) \rightarrow [0, \infty)$ . Assume that there is an increasing sequence  $(s_k)_{k \geq 1} \subset [0, \infty)$  such that  $s_k \rightarrow \infty$  and for each  $k$  the sequence  $(f_n(s_k))_{n \geq 1}$  has a limit that we denote  $f(s_k)$ . Let  $\ell = \lim_{k \rightarrow \infty} f(s_k)$  (the limit exists because  $(f(s_k))_{k \geq 1}$  is nondecreasing). Then:*

- (i) *For any sequence  $t_n \rightarrow \infty$  we have  $\liminf_{n \rightarrow \infty} f_n(t_n) \geq \ell$ .*
- (ii) *There exists a nondecreasing sequence  $(t_n^*)_{n \geq 1}$ ,  $t_n^* \rightarrow \infty$  such that  $f_n(t_n^*) \rightarrow \ell$ .*
- (iii) *If  $(t_n^*)_{n \geq 1}$  is as in (ii), for any sequence  $(t_n)_{n \geq 1}$  such that  $0 \leq t_n \leq t_n^*$  and  $t_n \rightarrow \infty$  we have  $f_n(t_n) \rightarrow \ell$ .*

*Proof.* (i) Fix  $\ell' < \ell$ . Choose  $k$  such that  $f(s_k) > \ell'$ . There is  $n_0$  such that  $f_n(s_k) > \ell'$  for all  $n \geq n_0$ . There is  $n_1 \geq n_0$  such that  $t_n \geq s_k$  for all  $n \geq n_1$ . Then for any  $n \geq n_1$  we have  $f_n(t_n) \geq f_n(s_k) > \ell'$  and we infer that  $\liminf_{n \rightarrow \infty} f_n(t_n) \geq \ell'$ . Since  $\ell' < \ell$  is arbitrary, (i) follows.

(ii) If  $\ell = \infty$ , it follows from (i) that any sequence  $t_n \rightarrow \infty$  satisfies (ii). Assume next that  $\ell$  is finite. Then  $0 \leq f(s_k) < \infty$  for all  $k$ . We choose inductively  $n_1, \dots, n_k, \dots \in \mathbf{N}$  such that  $n_k < n_{k+1}$  and

$$|f_n(s_k) - f(s_k)| < \frac{1}{k} \quad \text{for all } n \geq n_k.$$

If  $n_k \leq n < n_{k+1}$  put  $t_n^* = s_k$ . Then for  $n_k \leq n < n_{k+1}$  we have

$$|f_n(t_n^*) - \ell| = |f_n(s_k) - \ell| \leq |f_n(s_k) - f(s_k)| + |f(s_k) - \ell| < \frac{1}{k} + |f(s_k) - \ell| \rightarrow 0$$

as  $k \rightarrow \infty$ , hence  $t_n^*$  satisfies (ii).

Assertion (iii) is a simple consequence of (i) and (ii). □

**Lemma 2.3** *Let  $(q_n)_{n \geq 1}$ ,  $q_n : [0, \infty) \rightarrow \mathbf{R}$  be a sequence of nondecreasing functions such that  $(q_n(t))_{n \geq 1}$  is bounded for any fixed  $t \geq 0$ . Denote  $\tilde{q}(t) = \limsup_{n \rightarrow \infty} q_n(t)$  and  $\alpha = \lim_{t \rightarrow \infty} \tilde{q}(t)$  (the last limit exists because  $\tilde{q}$  is nondecreasing).*

*For any  $\beta < \alpha$  there is a subsequence  $(q_{n_k})_{k \geq 1}$  and a nondecreasing function  $q : [0, \infty) \rightarrow \mathbf{R}$  such that  $q_{n_k}(t) \rightarrow q(t)$  as  $k \rightarrow \infty$  for any  $t \geq 0$  and  $\lim_{t \rightarrow \infty} q(t) > \beta$ .*

*Proof.* Fix  $\beta < \alpha$ . Then fix  $t_0 > 0$  such that  $\tilde{q}(t_0) > \beta$ . There is an increasing mapping  $j_0 : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that  $q_{j_0(n)}(t_0) \rightarrow \tilde{q}(t_0) > \beta$ . We may write  $\mathbf{Q}_+ \setminus \{t_0\} = \{t_1, t_2, \dots, t_n, \dots\}$ , where  $t_i \neq t_j$  if  $i \neq j$ .

Inductively we construct increasing mappings  $j_1, j_2, \dots, j_n, \dots : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that  $q_{j_0(j_1(n))}(t_1)$  converges to a limit denoted  $q(t_1)$ ,  $q_{j_0(j_1(j_2(n))))(t_2)$  converges to a limit denoted  $q(t_2)$  and so on,  $q_{j_0(j_1(j_2(\dots j_k(n) \dots)))}(t_k)$  converges as  $n \rightarrow \infty$  to a limit denoted  $q(t_k)$  for any  $k \geq 1$ . Let  $j(n) = j_0(j_1(\dots(j_n(n)) \dots))$  and  $q(t_0) = \tilde{q}(t_0)$ . It is clear that  $j : \mathbf{N}^* \rightarrow \mathbf{N}^*$  is increasing,  $q_{j(n)}(t) \rightarrow q(t)$  as  $n \rightarrow \infty$  for all  $t \in \mathbf{Q}_+ \cup \{t_0\}$  and  $q$  is a nondecreasing mapping defined on  $\mathbf{Q}_+ \cup \{t_0\}$ .

We extend  $q$  to a nondecreasing mapping on  $[0, \infty)$  by defining  $q(t) = \sup\{q(s) \mid s \in \mathbf{Q}_+ \cup \{t_0\}, s \leq t\}$ . We claim that  $q_{j(n)}(t) \rightarrow q(t)$  as  $n \rightarrow \infty$  at any  $t > 0$  where  $q$  is continuous. Fix  $t > 0$  such that  $q$  is continuous at  $t$ . Fix  $\varepsilon > 0$ . There are  $r, s \in \mathbf{Q}_+$  such that  $r < t < s$  and  $q(t) - \frac{\varepsilon}{2} < q(r) \leq q(t) \leq q(s) < q(t) + \frac{\varepsilon}{2}$ . There is  $n_\varepsilon \in \mathbf{N}$  such that  $q_{j(n)}(r) > q(r) - \frac{\varepsilon}{2}$  and  $q_{j(n)}(s) < q(s) + \frac{\varepsilon}{2}$  for all  $n \geq n_\varepsilon$ . Hence

$$q(t) - \varepsilon < q(r) - \frac{\varepsilon}{2} < q_{j(n)}(r) \leq q_{j(n)}(t) \leq q_{j(n)}(s) < q(s) + \frac{\varepsilon}{2} < q(t) + \varepsilon$$

for all  $n \geq n_\varepsilon$ . Since  $\varepsilon$  is arbitrary this implies  $q_{j(n)}(t) \rightarrow q(t)$ . Moreover,  $q(t_0) = \tilde{q}(t_0) > \beta$ , hence  $q(t) \geq q(t_0) > \beta$  for all  $t \geq t_0$ .

There are at most countably many points where  $q$  is discontinuous, say  $d_1, d_2, \dots, d_k, \dots$ . Using again the diagonal extraction procedure we construct an increasing mapping  $\kappa : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that  $(q_{j(\kappa(n))}(d_\ell))_{n \geq 1}$  converges for any  $\ell \geq 1$ . Then we redefine  $q(d_\ell) = \lim_{n \rightarrow \infty} q_{j(\kappa(n))}(d_\ell)$ . It is obvious that  $q$  and the subsequence  $(q_{j(\kappa(n))})_{n \geq 1}$  satisfy the conclusion of Lemma 2.3 □

**Remark 2.4** Lemma 2.3 is a refinement of Helly's Lemma. It is optimal in the following sense: we may construct a sequence of nondecreasing functions  $q_n : [0, \infty) \rightarrow [0, 1]$  such that  $\lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n(t) \right) = 1$  and for any subsequence  $(q_{n_k})_{k \geq 1}$  such that there exists  $t_0 > 0$  satisfying  $\lim_{k \rightarrow \infty} q_{n_k}(t_0) > 0$ , there holds  $\lim_{t \rightarrow \infty} \left( \limsup_{k \rightarrow \infty} q_{n_k}(t) \right) < 1$ .

### 3 Proof of Lemma 1.3

Fix an increasing function  $\varphi: [0, \infty) \rightarrow [0, \infty)$  such that  $\varphi(t) \leq \frac{t}{2}$  for all  $t$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . We prove first a relaxed version of Lemma 1.3 which asserts only that  $(\mu_{n_k|B(x_k, R_k)})_{k \geq 1}$  concentrates (not necessarily around  $(x_k)_{k \geq 1}$ ) and

$$(3.1) \quad \mu_{n_k}(B(x_k, R_k) \setminus B(x_k, \varphi(R_k))) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows from Lemma 2.2 (iii) that  $q_n(\varphi(t_n)) \rightarrow \alpha$ . For each  $n$  there is  $x_n^0 \in \mathcal{X}$  such that  $\mu_n(B(x_n^0, \varphi(t_n))) > q_n(\varphi(t_n)) - \frac{1}{n}$ . It is clear that  $q_n(t_n) \geq \mu_n(B(x_n^0, t_n)) \geq \mu_n(B(x_n^0, \varphi(t_n)))$ , therefore

$$(3.2) \quad \mu_n(B(x_n^0, \varphi(t_n))) \rightarrow \alpha \quad \text{and} \quad \mu_n(B(x_n^0, t_n) \setminus B(x_n^0, \varphi(t_n))) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Denote by  $q_n^1(t)$  the concentration function of  $\mu_n|_{B(x_n^0, t_n)}$ . For each  $t \geq 0$  let  $\tilde{q}^1(t) = \limsup_{n \rightarrow \infty} q_n^1(t)$ . Using Lemma 2.3 we see that there are a nondecreasing function  $q^1: [0, \infty) \rightarrow [0, M]$  and an increasing mapping  $j_1: \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that

$$(3.3) \quad q_{j_1(n)}^1(t) \rightarrow q^1(t) \quad \text{for all } t > 0 \text{ and } \lim_{t \rightarrow \infty} q^1(t) > \lim_{t \rightarrow \infty} \tilde{q}^1(t) - \frac{\alpha}{2^2}.$$

Let  $\alpha_1 = \lim_{t \rightarrow \infty} q^1(t)$ . Since  $q_n^1 \leq q_n$ , it is clear that  $q^1 \leq q$  and  $0 \leq \alpha_1 \leq \alpha$ . If  $\alpha_1 = \alpha$  it follows from (3.2) and Remark 2.1 (iv) that  $\left(\mu_{j_1(n)}|_{B(x_{j_1(n)}^0, t_{j_1(n)})}\right)_{n \geq 1}$  concentrates, hence the conclusion of Lemma 1.3 (relaxed) is satisfied by  $(\mu_{j_1(n)})_{n \geq 1}$ ,  $(x_{j_1(n)}^0)_{n \geq 1}$  and  $(t_{j_1(n)})_{n \geq 1}$ .

Assume that  $\alpha_1 < \alpha$ . By Lemma 2.2 (ii)-(iii) there is an increasing sequence  $t_n^1 \rightarrow \infty$  such that  $t_n^1 < \frac{1}{4}t_{j_1(n)}$  and  $q_{j_1(n)}^1(t_n^1) \rightarrow \alpha_1$ . Then using the fact that  $q_{j_1(n)}(t_{j_1(n)}) \rightarrow \alpha$ , (1.1) and Lemma 2.2 (iii) we get

$$q_{j_1(n)}(t_n^1) \rightarrow \alpha \quad \text{and} \quad q_{j_1(n)}(\varphi(t_n^1)) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

For each  $n$  there is  $x_n^1 \in \mathcal{X}$  such that

$$(3.4) \quad \mu_{j_1(n)}(B(x_n^1, \varphi(t_n^1))) > q_{j_1(n)}(\varphi(t_n^1)) - \frac{1}{2j_1(n)}.$$

It is clear that  $\mu_{j_1(n)}(B(x_n^1, \varphi(t_n^1))) \leq \mu_{j_1(n)}(B(x_n^1, t_n^1)) \leq q_{j_1(n)}(t_n^1)$ , hence

$$(3.5) \quad \mu_{j_1(n)}(B(x_n^1, \varphi(t_n^1))) \rightarrow \alpha \quad \text{and} \quad \mu_{j_1(n)}(B(x_n^1, t_n^1)) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

We claim that there is  $n_1 \in \mathbf{N}^*$  such that for all  $n \geq n_1$ ,

$$(3.6) \quad B(x_{j_1(n)}^0, \varphi(t_{j_1(n)})) \cap B(x_n^1, t_n^1) = \emptyset.$$

To see this we argue by contradiction and assume that the balls intersect. Recalling that  $\varphi(t_{j_1(n)}) \leq \frac{1}{2}t_{j_1(n)}$  and  $t_n^1 < \frac{1}{4}t_{j_1(n)}$ , we have

$$B(x_n^1, t_n^1) \subset B(x_{j_1(n)}^0, \varphi(t_{j_1(n)}) + 2t_n^1) \subset B(x_{j_1(n)}^0, t_{j_1(n)})$$

and using (3.4) we get

$$q_{j_1(n)}^1(t_n^1) \geq \mu_{j_1(n)|B(x_{j_1(n)}^0, t_{j_1(n)})}(B(x_n^1, t_n^1)) = \mu_{j_1(n)}(B(x_n^1, t_n^1)) > q_{j_1(n)}(\varphi(t_n^1)) - \frac{1}{2j_1(n)}.$$

Since  $q_{j_1(n)}^1(t_n^1) \rightarrow \alpha_1 < \alpha$  and  $q_{j_1(n)}(\varphi(t_n^1)) \rightarrow \alpha$  as  $n \rightarrow \infty$  we see that the last inequality cannot hold for  $n$  sufficiently large. The claim (3.6) is thus proven.

Replacing  $j_1$  by  $j_1(n_1 + \cdot)$ , we may assume that (3.6) holds for any  $n \in \mathbf{N}^*$ .

Let  $q_n^2(t)$  be the concentration function of  $\mu_{j_1(n)|B(x_n^1, t_n^1)}$ . Let  $\tilde{q}^2(t) = \limsup_{n \rightarrow \infty} q_n^2(t)$ . As previously, by Lemma 2.3 there are a nondecreasing function  $q^2 : [0, \infty) \rightarrow [0, M]$  and an increasing mapping  $j_2 : \mathbf{N}^* \rightarrow \mathbf{N}^*$  such that

$$(3.7) \quad q_{j_2(n)}^2(t) \rightarrow q^2(t) \quad \text{for all } t > 0 \text{ and } \lim_{t \rightarrow \infty} q^2(t) > \lim_{t \rightarrow \infty} \tilde{q}^2(t) - \frac{\alpha}{2^3}.$$

Let  $\alpha_2 = \lim_{t \rightarrow \infty} q^2(t)$ . Since  $q_n^2 \leq q_{j_1(n)}$ , it is clear that  $q^2 \leq q$  and  $0 \leq \alpha_2 \leq \alpha$ . If  $\alpha_2 = \alpha$  it follows from (3.5) and Remark 2.1 (iv) that  $\left( \mu_{j_1(j_2(n))|B(x_{j_2(n)}^1, t_{j_2(n)}^1)} \right)_{n \geq 1}$  concentrates, hence  $(\mu_{j_1(j_2(n))})_{n \geq 1}$ ,  $(x_{j_2(n)}^1)_{n \geq 1}$  and  $(t_{j_2(n)}^1)_{n \geq 1}$  satisfy the conclusion of Lemma 1.3 (relaxed).

If  $\alpha_2 < \alpha$ , we continue as above. By Lemma 2.2 (ii)-(iii) there is an increasing sequence  $t_n^2 \rightarrow \infty$  such that  $t_n^2 < \frac{1}{4}t_{j_2(n)}^1$  and  $q_{j_2(n)}^2(t_n^2) \rightarrow \alpha_2$ . Since  $t_n^2 < \frac{1}{4}t_{j_2(n)}^1 < \frac{1}{4^2}t_{j_1(j_2(n))}$ , Lemma 2.2 (iii) implies that

$$q_{j_1(j_2(n))}(t_n^2) \rightarrow \alpha \quad \text{and} \quad q_{j_1(j_2(n))}(\varphi(t_n^2)) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

For each  $n$  choose  $x_n^2 \in \mathcal{X}$  such that

$$\mu_{j_1(j_2(n))}(B(x_n^2, \varphi(t_n^2))) > q_{j_1(j_2(n))}(\varphi(t_n^2)) - \frac{1}{2^2 j_1(j_2(n))}.$$

Obviously,

$$q_{j_1(j_2(n))}(t_n^2) \geq \mu_{j_1(j_2(n))}(B(x_n^2, t_n^2)) \geq \mu_{j_1(j_2(n))}(B(x_n^2, \varphi(t_n^2))) > q_{j_1(j_2(n))}(\varphi(t_n^2)) - \frac{1}{2^2 j_1(j_2(n))},$$

hence

$$\mu_{j_1(j_2(n))}(B(x_n^2, t_n^2)) \rightarrow \alpha \quad \text{and} \quad \mu_{j_1(j_2(n))}(B(x_n^2, \varphi(t_n^2))) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

There is  $n_2 \in \mathbf{N}^*$  such that for all  $n \geq n_2$ ,

$$(3.8) \quad B(x_{j_1(j_2(n))}^0, \varphi(t_{j_1(j_2(n))})) \cap B(x_n^2, t_n^2) = \emptyset \quad \text{and} \quad B(x_{j_2(n)}^1, \varphi(t_{j_2(n)}^1)) \cap B(x_n^2, t_n^2) = \emptyset.$$

The proof of (3.8) is similar to the proof of (3.6) and we omit it.

Replacing  $j_2$  by  $j_2(\cdot + n_2)$  we may assume that (3.8) holds for any  $n \in \mathbf{N}^*$ .

We continue the process inductively. Suppose that we have constructed increasing mappings  $j_1, \dots, j_k : \mathbf{N}^* \rightarrow \mathbf{N}^*$  and for each  $i \in \{1, \dots, k\}$  we have an increasing sequence  $(t_n^i)_{n \geq 1}$ , a sequence of points  $(x_n^i)_{n \geq 1} \subset \mathcal{X}$  and nondecreasing functions  $q_n^i, q^i : [0, \infty) \rightarrow [0, M]$  with the following properties:

$$(P1) \quad t_n^i \rightarrow \infty \text{ as } n \rightarrow \infty, \quad t_n^i < \frac{1}{4}t_{j_i(n)}^{i-1} \text{ for all } n \in \mathbf{N}^* \text{ and } i \in \{1, \dots, k\}, \text{ where } t_n^0 = t_n.$$

As an obvious consequence of (P1) we have

$$(3.9) \quad t_n^k < \frac{1}{4}t_{j_k(n)}^{k-1} < \frac{1}{4^2}t_{j_{k-1}(j_k(n))}^{k-2} < \dots < \frac{1}{4^\ell}t_{j_{k-\ell+1}(\dots(j_k(n))\dots)}^{k-\ell} < \dots < \frac{1}{4^k}t_{j_1(j_2(\dots(j_k(n))\dots))}.$$

$$(P2) \quad q_n^i(t) \text{ is the concentration function of } \mu_{j_1(j_2(\dots(j_{i-1}(n))\dots))|B(x_n^{i-1}, t_n^{i-1})},$$



(P3)  $q_{j_i(n)}^i(t) \rightarrow q^i(t)$  as  $n \rightarrow \infty$  and  $q^i(t) \rightarrow \alpha_i$  as  $t \rightarrow \infty$ , where

$$(3.10) \quad \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^i(t) \right) - \frac{\alpha}{2^{i+1}} < \alpha_i < \alpha.$$

(P4)  $q_{j_i(n)}^i(t_n^i) \rightarrow \alpha_i$ ,  $q_{j_i(n)}^i(\varphi(t_n^i)) \rightarrow \alpha_i$  as  $n \rightarrow \infty$  and

$$(3.11) \quad \mu_{j_1(\dots(j_i(n))\dots)}(B(x_n^i, \varphi(t_n^i))) > q_{j_1(\dots(j_i(n))\dots)}(\varphi(t_n^i)) - \frac{1}{2^i j_1(\dots(j_i(n))\dots)}.$$

(P5) We have for all  $n \geq 1$ ,  $\ell = 1, \dots, k$  and  $i = 1, \dots, \ell$ ,

$$(3.12) \quad B\left(x_{j_i(j_{i+1}(\dots(j_\ell(n)\dots)))}^{i-1}, \varphi(t_{j_i(j_{i+1}(\dots(j_\ell(n)\dots)))}^{i-1})\right) \cap B(x_n^\ell, t_n^\ell) = \emptyset.$$

Let  $q_n^{k+1}(t)$  be the concentration function of  $\mu_{j_1(\dots(j_k(n))\dots)}|_{B(x_n^k, t_n^k)}$ . We have

$$\begin{aligned} q_{j_1(\dots(j_k(n))\dots)}(t_{j_1(\dots(j_k(n))\dots)}^k) &\geq q_{j_1(\dots(j_k(n))\dots)}(t_n^k) && \text{by (3.9)} \\ &\geq \mu_{j_1(\dots(j_k(n))\dots)}(B(x_n^k, t_n^k)) \geq \mu_{j_1(\dots(j_k(n))\dots)}(B(x_n^k, \varphi(t_n^k))) \\ &\geq q_{j_1(\dots(j_k(n))\dots)}(\varphi(t_n^k)) - \frac{1}{2^k j_1(\dots(j_k(n))\dots)} && \text{by (3.11).} \end{aligned}$$

Hence using (1.1), the fact that  $q_n(t_n) \rightarrow \alpha$  and Lemma 2.2 (iii) we get

$$\mu_{j_1(\dots(j_k(n))\dots)}(B(x_n^k, t_n^k)) \rightarrow \alpha \quad \text{and} \quad \mu_{j_1(\dots(j_k(n))\dots)}(B(x_n^k, \varphi(t_n^k))) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

It is clear that for all  $t \geq t_n^k$  we have  $q_n^{k+1}(t) = \mu_{j_1(\dots(j_k(n))\dots)}(B(x_n^k, t_n^k))$  and the last quantity tends to  $\alpha$  as  $n \rightarrow \infty$ .

Let  $\tilde{q}^{k+1}(t) = \limsup_{n \rightarrow \infty} q_n^{k+1}(t)$ . As above, using Lemma 2.3 we find an increasing mapping  $j_{k+1} : \mathbf{N}^* \rightarrow \mathbf{N}^*$  and a nondecreasing function  $q^{k+1} : [0, \infty) \rightarrow [0, M]$  such that

$$(3.13) \quad q_{j_{k+1}(n)}^{k+1}(t) \rightarrow q^{k+1}(t) \quad \text{for all } t > 0 \text{ and } \lim_{t \rightarrow \infty} q^{k+1}(t) > \lim_{t \rightarrow \infty} \tilde{q}^{k+1}(t) - \frac{\alpha}{2^{k+2}}.$$

Let  $\alpha_{k+1} = \lim_{t \rightarrow \infty} q^{k+1}(t)$ . From the above it is clear that  $\alpha_{k+1} \in [0, \alpha]$ . If  $\alpha_{k+1} = \alpha$  the conclusion of Lemma 1.3 (relaxed) holds for the subsequences  $(\mu_{j_1(\dots(j_{k+1}(n))\dots)})_{n \geq 1}$ ,  $(x_{j_{k+1}(n)}^k)_{n \geq 1}$  and  $(t_{j_{k+1}(n)}^k)_{n \geq 1}$ .

If  $\alpha_{k+1} < \alpha$  we perform another step in our extraction process. By Lemma 2.2 (ii)-(iii) there is an increasing sequence  $t_n^{k+1} \rightarrow \infty$  such that

$$(3.14) \quad t_n^{k+1} < \frac{1}{4} t_{j_{k+1}(n)}^k \quad \text{and} \quad q_{j_{k+1}(n)}^{k+1}(t_n^{k+1}) \rightarrow \alpha_{k+1} \quad \text{as } n \rightarrow \infty.$$

Since  $t_n^{k+1} < \frac{1}{4} t_{j_{k+1}(n)}^k < \frac{1}{4^{k+1}} t_{j_1(\dots(j_{k+1}(n))\dots)}$ , Lemma 2.2 (iii) implies that

$$q_{j_1(\dots(j_{k+1}(n))\dots)}(t_n^{k+1}) \rightarrow \alpha \quad \text{and} \quad q_{j_1(\dots(j_{k+1}(n))\dots)}(\varphi(t_n^{k+1})) \rightarrow \alpha \quad \text{as } n \rightarrow \infty.$$

For each  $n$  we choose  $x_n^{k+1} \in \mathcal{X}$  such that

$$\mu_{j_1(\dots(j_{k+1}(n))\dots)}(B(x_n^{k+1}, \varphi(t_n^{k+1}))) > q_{j_1(\dots(j_{k+1}(n))\dots)}(\varphi(t_n^{k+1})) - \frac{1}{2^{k+2} j_1(\dots(j_{k+1}(n))\dots)}.$$

Since

$$q_{j_1(\dots(j_{k+1}(n))\dots)}(t_n^{k+1}) \geq \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, t_n^{k+1}) \right) \geq \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, \varphi(t_n^{k+1})) \right),$$

we have

$$(3.15) \quad \begin{aligned} \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, t_n^{k+1}) \right) &\longrightarrow \alpha \quad \text{and} \\ \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, \varphi(t_n^{k+1})) \right) &\longrightarrow \alpha \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

We show that there is  $n_{k+1} \in \mathbf{N}$  such that for all  $n \geq n_{k+1}$ ,

$$(3.16) \quad \forall i \in \{1, \dots, k+1\}, \quad B \left( x_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}, \varphi(t_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}) \right) \cap B(x_n^{k+1}, t_n^{k+1}) = \emptyset.$$

Indeed, suppose that for some  $i \in \{1, \dots, k+1\}$  the intersection in (3.16) is not empty. Then

$$(3.17) \quad \begin{aligned} B(x_n^{k+1}, t_n^{k+1}) &\subset B \left( x_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}, \varphi(t_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}) + 2t_n^{k+1} \right) \\ &\subset B \left( x_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}, t_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1} \right) \end{aligned}$$

because  $\varphi(s) \leq \frac{s}{2}$  and  $t_n^{k+1} < \frac{1}{4}t_{j_i(\dots(j_{k+1}(n))\dots)}^{i-1}$ . If  $i \leq k$  we get

$$\begin{aligned} &q_{j_i(j_{i+1}(\dots(j_{k+1}(n))\dots))}^i \left( t_{j_{i+1}(\dots(j_{k+1}(n))\dots)}^i \right) \\ &\geq q_{j_i(j_{i+1}(\dots(j_{k+1}(n))\dots))}^i(t_n^{k+1}) \quad \text{by (3.9) and (3.14)} \\ &\geq \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left| B \left( x_{j_i(j_{i+1}(\dots(j_{k+1}(n))\dots))}^{i-1}, t_{j_i(j_{i+1}(\dots(j_{k+1}(n))\dots))}^{i-1} \right) \right| \left( B(x_n^{k+1}, t_n^{k+1}) \right) \\ &= \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, t_n^{k+1}) \right) \quad \text{by (3.17).} \end{aligned}$$

As  $n \longrightarrow \infty$  the right-hand side of the above inequality tends to  $\alpha$  by (3.15) and the left-hand side tends to  $\alpha_i < \alpha$  by (P4), hence the inequality may be true only for finitely many  $n$ 's.

If  $i = k+1$  we have  $B(x_n^{k+1}, t_n^{k+1}) \subset B \left( x_{j_{k+1}(n)}^k, t_{j_{k+1}(n)}^k \right)$ , hence

$$\begin{aligned} q_{j_{k+1}(n)}^{k+1}(t_n^{k+1}) &\geq \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left| B \left( x_{j_{k+1}(n)}^k, t_{j_{k+1}(n)}^k \right) \right| \left( B(x_n^{k+1}, t_n^{k+1}) \right) \\ &= \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, t_n^{k+1}) \right). \end{aligned}$$

Since  $q_{j_{k+1}(n)}^{k+1}(t_n^{k+1}) \longrightarrow \alpha_{k+1} < \alpha$  by (3.14) and  $\mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B(x_n^{k+1}, t_n^{k+1}) \right) \longrightarrow \alpha$  as  $n \longrightarrow \infty$  by (3.15), the above inequality cannot hold for  $n$  sufficiently large.

We conclude that there is  $n_{k+1} \in \mathbf{N}$  such that (3.16) holds for all  $n \geq n_{k+1}$ .

Replacing  $j_{k+1}$  by  $j_{k+1}(\cdot + n_{k+1})$  we may assume that (3.16) holds for all  $n \in \mathbf{N}^*$ .

The mappings  $j_1, \dots, j_{k+1}$ , the functions  $q_n^i$  and  $q^i$  and the sequences  $(x_n^i)_{n \geq 1}$ ,  $(t_n^i)_{n \geq 1}$  with  $i \in \{1, \dots, k+1\}$  satisfy the properties (P1) - (P5) above and this finishes our induction. We conclude that at step  $k$  either we find a subsequence of  $(\mu_n)_{n \geq 1}$  and a sequence of balls satisfying the conclusion of Lemma 1.3 (relaxed), or we are able to complete the step  $k+1$ .

We claim that the above process has to stop after a finite number of steps. Indeed, assume that we have completed  $k$  steps. It follows immediately from (3.12) that for each  $n \in \mathbf{N}^*$  the balls

$$B \left( x_{j_1(\dots(j_k(n))\dots)}^0, \varphi(t_{j_1(\dots(j_k(n))\dots)}^0) \right), \dots, B \left( x_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell, \varphi(t_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell) \right), \dots, B(x_n^k, t_n^k)$$

are disjoint, where  $\ell = 1, \dots, k-1$ . From the choice of  $x_n^0$ , (3.4) and (3.11) we have

$$\begin{aligned} & \mu_{j_1(\dots(j_k(n))\dots)} \left( B \left( x_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell, \varphi(t_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell) \right) \right) \\ & > q_{j_1(\dots(j_k(n))\dots)} \left( \varphi(t_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell) \right) - \frac{1}{2^\ell j_1(\dots(j_k(n))\dots)} \quad \text{for } \ell = 0, 1, \dots, k-1, \quad \text{and} \\ & \mu_{j_1(\dots(j_k(n))\dots)} \left( B(x_n^k, \varphi(t_n^k)) \right) > q_{j_1(\dots(j_k(n))\dots)} \left( \varphi(t_n^k) \right) - \frac{1}{2^k j_1(\dots(j_k(n))\dots)}. \end{aligned}$$

Since the balls are disjoint, summing up the above inequalities we get

$$\begin{aligned} & \mu_{j_1(\dots(j_k(n))\dots)}(\mathcal{X}) \\ & \geq \sum_{\ell=0}^{k-1} q_{j_1(\dots(j_k(n))\dots)} \left( \varphi(t_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell) \right) + q_{j_1(\dots(j_k(n))\dots)} \left( \varphi(t_n^k) \right) - \frac{2}{j_1(\dots(j_k(n))\dots)}. \end{aligned}$$

We remember that  $\varphi(t_{j_{\ell+1}(\dots(j_k(n))\dots)}^\ell) \rightarrow \infty$  as  $n \rightarrow \infty$  for each  $\ell$ , then we take the lim sup in the above inequality and use Lemma 2.2 (i) to get  $M \geq (k+1)\alpha$ . This implies that  $k$  has to remain bounded, which means that we may perform only a finite number of steps in the previous extraction process. The relaxed version of Lemma 3.6 is thus proven.

It remains to show that the subsequences may be choosen in such a way that  $(\mu_{n_k})_{k \geq 1}$  concentrates around  $(x_k)_{k \geq 1}$ .

Let  $\varphi(t) = \frac{t}{12}$ . There exist a subsequence  $(\mu_{n_k})_{k \geq 1}$ , a sequence  $(z_k)_{k \geq 1} \subset \mathcal{X}$  and an increasing sequence  $\tilde{R}_k \rightarrow \infty$  satisfying the relaxed version of Lemma 1.3 with  $z_k, \tilde{R}_k$  instead of  $x_k$  and  $R_k$ , respectively. In particular, the sequence of measures  $(\mu_{n_k}|_{B(z_k, \tilde{R}_k)})_{k \geq 1}$  concentrates around some sequence of points  $(x_k)_{k \geq 1}$ .

Using Remark 2.1 (i) we get

$$\mu_{n_k} \left( B(z_k, \tilde{R}_k) \setminus B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) = \mu_{n_k}|_{B(z_k, \tilde{R}_k)} \left( \mathcal{X} \setminus B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and this implies  $\mu_{n_k} \left( B \left( z_k, \tilde{R}_k \right) \cap B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) \rightarrow \alpha$ . On the other hand we have

$$\begin{aligned} & \mu_{n_k} \left( B \left( z_k, \frac{\tilde{R}_k}{12} \right) \right) \geq \mu_{n_k} \left( B \left( z_k, \frac{\tilde{R}_k}{12} \right) \cap B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) \\ & \geq \mu_{n_k} \left( B \left( z_k, \tilde{R}_k \right) \cap B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) - \mu_{n_k} \left( B \left( z_k, \tilde{R}_k \right) \setminus B \left( z_k, \frac{\tilde{R}_k}{12} \right) \right). \end{aligned}$$

Using (3.1) we infer that  $\mu_{n_k} \left( B \left( z_k, \frac{\tilde{R}_k}{12} \right) \cap B \left( x_k, \frac{\tilde{R}_k}{12} \right) \right) \rightarrow \alpha$ . In particular, for all  $k$  sufficiently large we have  $B \left( x_k, \frac{\tilde{R}_k}{12} \right) \cap B \left( z_k, \frac{\tilde{R}_k}{12} \right) \neq \emptyset$ , hence  $d(x_k, z_k) \leq \frac{\tilde{R}_k}{6}$ . Let  $R_k = \frac{1}{2}\tilde{R}_k$ . Then

$$B \left( z_k, \frac{\tilde{R}_k}{12} \right) \subset B \left( x_k, \frac{\tilde{R}_k}{4} \right) = B \left( x_k, \frac{R_k}{2} \right) \subset B(x_k, R_k) \subset B(z_k, \tilde{R}_k).$$

Hence  $\mu_{n_k}(B(x_k, R_k)) \rightarrow \alpha$  as  $k \rightarrow \infty$ , the sequence of measures  $(\mu_{n_k}|_{B(x_k, R_k)})_{k \geq 1}$  concentrates around  $(x_k)_{k \geq 1}$  (cf. Remark 2.1 (i)) and Lemma 1.3 is proven.  $\square$

## 4 Proof of Theorem 1.5

Let  $q_n(t)$  be the concentration function of  $\mu_n$  and let

$$\alpha_0 := \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n(t) \right).$$

If  $\alpha_0 = 0$  then  $(\mu_n)_{n \geq 1}$  is a vanishing sequence and the conclusion of Theorem 1.5 holds.

From now on we assume that  $\alpha_0 > 0$  and we use the extraction procedure described below.

*Step 1.* Let  $(\nu_n)_{n \geq 1}$  be a sequence of positive Borel measures on  $\mathcal{X}$  with concentration functions  $q_n$  such that  $\nu_n(\mathcal{X})$  is bounded and

$$(4.1) \quad \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n(t) \right) = \alpha > 0.$$

Fix  $s > 0$ ,  $\varepsilon_1, \varepsilon_2 > 0$  and an increasing sequence of positive numbers  $(t_n)_{n \geq 1}$ ,  $t_n \rightarrow \infty$ .

We show that there exist an increasing mapping  $j : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a sequence of points  $(x_n)_{n \geq 1} \subset \mathcal{X}$  and an increasing sequence  $(R_n)_{n \geq 1}$ ,  $R_n \rightarrow \infty$  with the following properties:

- (p1)  $q_{j(n)}(s) < 2\alpha$  for any  $n$ ,
- (p2)  $R_n \leq t_{j(n)}$  for all  $n$ ,
- (p3)  $0 < \nu_{j(n)}(B(x_n, \varphi(R_n))) \rightarrow \alpha'$  as  $n \rightarrow \infty$ , where  $\alpha' > \alpha - \varepsilon_1$ ,
- (p4)  $\nu_{j(n)}(B(x_n, 2R_n) \setminus B(x_n, \varphi(R_n))) < \frac{\varepsilon_2}{2^n}$ ,
- (p5)  $\left( \nu_{j(n)}|_{B(x_n, 2R_n)} \right)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$ .

By (4.1) we have  $\limsup_{n \rightarrow \infty} q_n(t) \leq \alpha$  for all  $t > 0$ . Eliminating a finite number of terms we may assume that  $q_n(s) < 2\alpha$  for all  $n$ .

Using Lemma 2.3 we find an increasing mapping  $\kappa_1 : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a nondecreasing function  $q : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{n \rightarrow \infty} q_{\kappa_1(n)}(t) = q(t)$  for any  $t > 0$  and  $\lim_{t \rightarrow \infty} q(t) > \alpha - \varepsilon_1$ .

Let  $\alpha' = \lim_{t \rightarrow \infty} q(t)$ . We may assume that  $\lim_{n \rightarrow \infty} q_{\kappa_1(n)}(t_{\kappa_1(n)}) = \alpha'$  (otherwise we replace  $(t_{\kappa_1(n)})_{n \geq 1}$  by a sequence  $(\tilde{t}_{\kappa_1(n)})_{n \geq 1}$  which has this property and satisfies  $\tilde{t}_{\kappa_1(n)} \leq t_{\kappa_1(n)}$ ; the existence of  $(\tilde{t}_{\kappa_1(n)})_{n \geq 1}$  is guaranteed by Lemma 2.2 (ii)-(iii)).

Using Lemma 1.3 for the sequences  $(\nu_{\kappa_1(n)})_{n \geq 1}$  and  $(t_{\kappa_1(n)})_{n \geq 1}$  we find an increasing mapping  $\kappa_2 : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a sequence of points  $(y_n)_{n \geq 1} \subset \mathcal{X}$  and a sequence  $R_n^*$  such that  $R_n^* \leq t_{\kappa_1(\kappa_2(n))}$ ,  $R_n^* \rightarrow \infty$ ,  $\nu_{\kappa_1(\kappa_2(n))}(B(y_n, \varphi(R_n^*/2))) \rightarrow \alpha'$ ,

$$(4.2) \quad \beta_n := \nu_{\kappa_1(\kappa_2(n))}(B(y_n, R_n^*) \setminus B(y_n, \varphi(R_n^*/2))) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and the sequence  $\left( \nu_{\kappa_1(\kappa_2(n))}|_{B(y_n, R_n^*)} \right)_{n \geq 1}$  concentrates.

We construct  $\kappa_3 : \mathbf{N}^* \rightarrow \mathbf{N}^*$  inductively as follows:  $\kappa_3(1)$  is the first integer satisfying  $\beta_{\kappa_3(1)} < \frac{1}{2}$  (where  $\beta_n$  is defined in (4.2)). If  $\kappa_3(1), \dots, \kappa_3(n)$  have been constructed, we define  $\kappa_3(n+1)$  as the first integer  $k_{n+1}$  verifying  $k_{n+1} > \kappa_3(n)$ ,  $\beta_{k_{n+1}} < \frac{1}{2^{n+1}}$  and  $R_{\kappa_3(n)}^* < R_{k_{n+1}}^*$ .

Let  $j = \kappa_1 \circ \kappa_2 \circ \kappa_3$ ,  $x_n = y_{\kappa_3(n)}$  and  $R_n = \frac{1}{2} R_{\kappa_3(n)}^*$ . Then  $j$ ,  $(x_n)_{n \geq 1}$  and  $(R_n)_{n \geq 1}$  have the properties (p1)–(p5) listed above.

We apply step 1 to  $(\mu_n)_{n \geq 1}$  with  $s = 1$ ,  $\varepsilon_1 = \frac{\alpha_0}{2}$ ,  $\varepsilon_2 = 1$  and we get an increasing mapping  $j_1 : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a sequence of points  $(z_n^1)_{n \geq 1} \subset \mathcal{X}$  and an increasing sequence  $(R_n^1)_{n \geq 1}$  with the properties (p1)–(p5) above. Denote

$$m_1 = \lim_{n \rightarrow \infty} \mu_{j_1(n)}(B(z_n^1, \varphi(R_n^1))) \in \left( \frac{\alpha_0}{2}, \alpha_0 \right].$$

Let  $\mu_n^1 = \mu_{j_1(n)}|_{\mathcal{X} \setminus B(z_n^1, R_n^1)}$  and let  $q_n^1$  be the concentration function of  $\mu_n^1$ . Denote

$$(4.3) \quad \alpha_1 = \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^1(t) \right).$$

Since  $q_n^1(t) \leq q_{j_1(n)}(t)$  it is clear that  $0 \leq \alpha_1 \leq \alpha_0$ . If  $\alpha_1 = 0$  conclusion (i) in Theorem 1.5 is satisfied (with  $k = 1$ ) by  $j_1$  and the balls  $B(z_n^1, R_n^1)$ . If  $\alpha_1 > 0$  we proceed to step 2.

*Step 2.* We apply step 1 to the sequence  $(\mu_n^1)_{n \geq 1}$  with  $s = 2$ ,  $\varepsilon_1 = \frac{\alpha_1}{2}$ ,  $\varepsilon_2 = \frac{1}{2}$  and  $t_n = \frac{1}{2}R_n^1$ . We obtain an increasing mapping  $j_2 : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a sequence of points  $(z_n^2)_{n \geq 1} \subset \mathcal{X}$ , and an increasing sequence  $(R_n^2)_{n \geq 1} \subset (0, \infty)$  such that

- $R_n^2 \rightarrow \infty$  and  $R_n^2 \leq \frac{1}{2}R_{j_2(n)}^1$ ,
- $\mu_{j_2(n)}^1(B(z_n^2, \varphi(R_n^2))) \rightarrow m_2$ , where  $\frac{\alpha_1}{2} < m_2 \leq \alpha_1$ ,
- $\mu_{j_2(n)}^1(B(z_n^2, 2R_n^2) \setminus B(z_n^2, \varphi(R_n^2))) < \frac{1}{2^{n+1}}$ .

We show that

$$(4.4) \quad B(z_n^2, R_n^2) \cap B(z_{j_2(n)}^1, R_{j_2(n)}^1) = \emptyset$$

for all  $n$  sufficiently large. Indeed, if the two balls intersect we have

$$B(z_n^2, R_n^2) \subset B(z_{j_2(n)}^1, R_{j_2(n)}^1 + 2R_n^2) \subset B(z_{j_2(n)}^1, 2R_{j_2(n)}^1),$$

hence

$$\mu_{j_2(n)}^1(B(z_n^2, R_n^2)) \leq \mu_{j_2(n)}^1(B(z_{j_2(n)}^1, 2R_{j_2(n)}^1)) = \mu_{j_1(j_2(n))}^1(B(z_{j_2(n)}^1, 2R_{j_2(n)}^1) \setminus B(z_{j_2(n)}^1, R_{j_2(n)}^1)).$$

The left hand side in the above inequality tends to  $m_2 > 0$  and the right hand side tends to 0 as  $n \rightarrow \infty$ , consequently there is  $n_1 \in \mathbf{N}$  such that the inequality cannot hold for  $n \geq n_1$ . Therefore (4.4) is true for  $n \geq n_1$ . Replacing  $j_2$  by  $j_2(\cdot + n_1)$  we assume from now on that (4.4) holds for all  $n \in \mathbf{N}^*$ .

Let

$$\mu_n^2 = \mu_{j_2(n)}^1|_{\mathcal{X} \setminus B(z_n^2, R_n^2)} = \mu_{j_1(j_2(n))}^1|_{\mathcal{X} \setminus (B(z_{j_2(n)}^1, R_{j_2(n)}^1) \cup B(z_n^2, R_n^2))}.$$

Let  $q_n^2$  be the concentration function of  $\mu_n^2$  and let

$$\alpha_2 = \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^2(t) \right).$$

Since  $q_n^2 \leq q_{j_2(n)}^1$  we have obviously  $0 \leq \alpha_2 \leq \alpha_1$ . If  $\alpha_2 = 0$  conclusion (i) in Theorem 1.5 is satisfied with  $k = 2$ ,  $j = j_1 \circ j_2$ ,  $x_n^1 = z_{j_2(n)}^1$ ,  $x_n^2 = z_n^2$ ,  $r_n^1 = R_{j_2(n)}^1$ ,  $r_n^2 = R_n^2$ .

If  $\alpha_2 > 0$  we proceed to step 3, which consists in applying step 1 to the sequence  $(\mu_n^2)_{n \geq 1}$  with  $s = 3$ ,  $\varepsilon_1 = \frac{\alpha_2}{2}$ ,  $\varepsilon_2 = \frac{1}{2^2}$ ,  $t_n = \frac{1}{2}R_n^2$ , and so on.

We continue the above process inductively. Assume that we have completed  $k$  steps and we have found  $k$  increasing mappings  $j_1, \dots, j_k : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , increasing sequences of positive numbers  $(R_n^1)_{n \geq 1}, \dots, (R_n^k)_{n \geq 1}$  that tend to infinity, sequences of points  $(z_n^1)_{n \geq 1}, \dots, (z_n^k)_{n \geq 1} \subset \mathcal{X}$ ,  $k$  sequences of measures  $(\mu_n^i)_{n \geq 1}$  with concentration functions  $q_n^i$ ,  $i \in \{0, \dots, k-1\}$  (where  $\mu_n^0 = \mu_n$ ,  $q_n^0 = q_n$ ) and positive numbers  $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_{k-1}$ ,  $m_1, \dots, m_k$  satisfying the following properties:

$$(H1) \quad R_n^k \leq \frac{1}{2}R_{j_k(n)}^{k-1} \leq \frac{1}{2^2}R_{j_{k-1}(j_k(n))}^{k-2} \leq \dots \leq \frac{1}{2^{k-1}}R_{j_2(\dots(j_k(n))\dots)}^1.$$

$$(H2) \quad \begin{aligned} \mu_n^\ell &= \mu_{j_\ell(n)}^{\ell-1}|_{\mathcal{X} \setminus B(z_n^\ell, R_n^\ell)} \\ &= \mu_{j_1(\dots(j_\ell(n))\dots)}^0|_{\mathcal{X} \setminus (B(z_n^\ell, R_n^\ell) \cup B(z_{j_\ell(n)}^{\ell-1}, R_{j_\ell(n)}^{\ell-1}) \cup \dots \cup B(z_{j_2(\dots(j_\ell(n))\dots)}^1, R_{j_2(\dots(j_\ell(n))\dots)}^1))} \end{aligned}$$

$$(H3) \quad \lim_{t \rightarrow \infty} \left( \lim_{n \rightarrow \infty} q_n^{\ell-1}(t) \right) = \alpha_{\ell-1} \text{ and } \mu_{j_\ell(n)}^{\ell-1}(B(z_n^\ell, \varphi(R_n^\ell))) \rightarrow m_\ell, \text{ where } \frac{\alpha_{\ell-1}}{2} < m_\ell \leq \alpha_{\ell-1}.$$

(H4)  $\mu_{j_\ell(n)}^{\ell-1}(B(z_n^\ell, 2R_n^\ell) \setminus B(z_n^\ell, \varphi(R_n^\ell))) \leq \frac{1}{2^{n+\ell-1}}$  for all  $n$  and  $\ell = 1, \dots, k$ , where  $\mu_n^0 = \mu_n$ .

(H5) The balls

$$B(z_n^k, R_n^k), B(z_{j_k(n)}^{k-1}, R_{j_k(n)}^{k-1}), B(z_{j_{k-1}(j_k(n))}^{k-2}, R_{j_{k-1}(j_k(n))}^{k-2}), \dots, B(z_{j_2(\dots(j_k(n))\dots)}^1, R_{j_2(\dots(j_k(n))\dots)}^1)$$

are all disjoint.

(H6) The sequence  $\left( \mu_{j_1(\dots(j_\ell(n))\dots)} \Big|_{B(z_n^\ell, 2R_n^\ell)} \right)_{n \geq 1}$  concentrates around  $(z_n^\ell)_{n \geq 1}$ .

(H7)  $m_1 + \dots + m_k \leq M$ .

(H8)  $q_{j_\ell(n)}^{\ell-1}(\ell) \leq 2\alpha_{\ell-1}$  for all  $n$  and  $\ell = 1, \dots, k$ .

Let  $\mu_n^k = \mu_{j_k(n)}^{k-1} \Big|_{\mathcal{X} \setminus B(z_n^k, R_n^k)}$ . Denote by  $q_n^k$  the concentration function of  $\mu_n^k$  and let  $\alpha_k = \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^k(t) \right)$ . Property (H2) above implies that  $q_n^k \leq q_{j_k(n)}^{k-1}$ , hence  $0 \leq \alpha_k \leq \alpha_{k-1}$ . If  $\alpha_k = 0$  conclusion (i) in Theorem 1.5 is satisfied by  $j = j_1 \circ \dots \circ j_k$  and the balls in (H5) above.

If  $\alpha_k > 0$  we proceed to step  $k+1$  which consists in applying step 1 to the sequence of measures  $(\mu_n^k)_{n \geq 1}$  with  $s = k+1$ ,  $\varepsilon_1 = \frac{\alpha_k}{2}$ ,  $\varepsilon_2 = \frac{1}{2^k}$  and  $t_n = \frac{1}{2}R_n^k$ . We find an increasing mapping  $j_{k+1} : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , a sequence of points  $(z_n^{k+1})_{n \geq 1}$  and an increasing sequence of positive numbers  $(R_n^{k+1})_{n \geq 1}$  with the properties (p1)–(p5) in step 1. It is clear that (H1)–(H4), (H6) and (H8) hold with  $k+1$  instead of  $k$ . We claim that there is  $n_k \in \mathbf{N}$  such that for any  $n \geq n_k$  and any  $\ell \in \{1, \dots, k\}$  we have

$$(4.5) \quad B(z_n^{k+1}, R_n^{k+1}) \cap B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell) = \emptyset.$$

Indeed, if the intersection is not empty using (H1) we get

$$(4.6) \quad \begin{aligned} B(z_n^{k+1}, R_n^{k+1}) &\subset B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell + 2R_n^{k+1})) \\ &\subset B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, 2R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell)). \end{aligned}$$

From properties (p3) and (p4) in step 1 it follows that

$$\mu_{j_{k+1}(n)}^k(B(z_n^{k+1}, R_n^{k+1})) \longrightarrow m_{k+1} \in \left( \frac{\alpha_k}{2}, \alpha_k \right] \quad \text{as } n \longrightarrow \infty.$$

On the other hand using (H2) and (H4) we find

$$\begin{aligned} &\mu_{j_{k+1}(n)}^k \left( B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, 2R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell) \right) \\ &\leq \mu_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell \left( B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, 2R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell) \right) \\ &= \mu_{j_\ell(\dots(j_{k+1}(n))\dots)}^{\ell-1} \left( B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, 2R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell) \right. \\ &\quad \left. \setminus B(z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell) \right) \\ &\leq \frac{1}{2^{j_{\ell+1}(\dots(j_{k+1}(n))\dots) + \ell - 1}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

We conclude that the (4.6) cannot be true if  $n$  is sufficiently large and the claim (4.5) is proven. Replacing  $j_{k+1}$  by  $j_{k+1}(\cdot + n_k)$  we may assume that (4.5) holds for all  $n \in \mathbf{N}^*$ . In particular, we see that (H5) is satisfied at level  $k+1$ .

Let  $\mu_n^{k+1} = \mu_{j_{k+1}(n)}^k|_{\mathcal{X} \setminus B(z_n^{k+1}, R_n^{k+1})}$ . For  $\ell = 1, \dots, k+1$  we have

$$\begin{aligned} & \mu_{j_1(\dots(j_{k+1}(n))\dots)} \left( B \left( z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell \right) \right) \\ & \geq \mu_{j_\ell(\dots(j_{k+1}(n))\dots)}^{\ell-1} \left( B \left( z_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell, R_{j_{\ell+1}(\dots(j_{k+1}(n))\dots)}^\ell \right) \right) \longrightarrow m_\ell \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

Summing up the above inequalities for  $\ell = 1, \dots, k+1$ , taking into account (H5) and passing to the limit as  $n \longrightarrow \infty$  we get

$$M \geq \limsup_{n \rightarrow \infty} \mu_{j_1(\dots(j_{k+1}(n))\dots)}(\mathcal{X}) \geq m_1 + \dots + m_{k+1}.$$

Hence the properties (H1)–(H8) are verified at level  $k+1$  and the induction is complete.

If there is  $k \in \mathbf{N}$  such that  $\alpha_k = 0$ , the process stops after  $k$  steps and conclusion (i) in Theorem 1.5 is satisfied. Otherwise we have  $\alpha_k > 0$  for all  $k$ . In the latter case the series  $\sum_{k \geq 0} \alpha_k$  converges (because  $0 < \alpha_k < 2m_{k+1}$  and the series  $\sum_{k \geq 1} m_k$  converges in view of (H7)). Therefore  $\alpha_k \longrightarrow 0$  as  $k \longrightarrow \infty$ .

If  $\alpha_k > 0$  for all  $k$  we perform a diagonal extraction to get a subsequence satisfying conclusion (ii) in Theorem 1.5. Let  $j(n) = j_1(\dots(j_n(n))\dots)$ . For each  $n \in \mathbf{N}^*$  let

$$\begin{aligned} B(x_n^1, r_n^1) &= B \left( z_{j_2(\dots(j_n(n))\dots)}^1, R_{j_2(\dots(j_n(n))\dots)}^1 \right), \\ B(x_n^2, r_n^2) &= B \left( z_{j_3(\dots(j_n(n))\dots)}^2, R_{j_3(\dots(j_n(n))\dots)}^2 \right) \quad \text{and so on } \dots, \\ B(x_n^{n-1}, r_n^{n-1}) &= B \left( z_{j_n(n)}^{n-1}, R_{j_n(n)}^{n-1} \right), \\ B(x_n^n, r_n^n) &= B(z_n^n, R_n^n). \end{aligned}$$

It follows from (H5) that the balls are disjoint. For each fixed  $k$  the sequence  $(r_n^k)_{n \geq 1}$  is increasing and tends to infinity as  $n \longrightarrow \infty$ , and (H6) implies that  $\left( \mu_{j(n)}|_{B(x_n^k, r_n^k)} \right)_{n \geq k}$  concentrates around  $(x_n^k)_{n \geq k}$ . From (H3) and (H4) we get conclusion (ii) (b). By (H2) we have  $\mu_{j(n)}|_{\mathcal{X} \setminus \cup_{i=1}^\ell B(x_n^i, r_n^i)} = \mu_{j_{\ell+1}(\dots(j_n(n))\dots)}^\ell$  if  $\ell < n$  and  $\mu_{j(n)}|_{\mathcal{X} \setminus \cup_{i=1}^n B(x_n^i, r_n^i)} = \mu_n^n$ , and the concentration functions of these measures are  $q_{j_{\ell+1}(\dots(j_n(n))\dots)}^\ell$  and  $q_n^n$ , respectively. Since  $\lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_{j_{\ell+1}(\dots(j_n(n))\dots)}^\ell(t) \right) \leq \lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^n(t) \right) = \alpha_\ell$  and  $\alpha_\ell \longrightarrow 0$  as  $\ell \longrightarrow \infty$ , (ii) (c) follows. From (H2) and (H8) we get  $q_n^n(n) \leq q_{j_n(n)}^{n-1}(n) < 2\alpha_{n-1}$ , hence  $q_n^n(n) \longrightarrow 0$  as  $n \longrightarrow \infty$ . Then using Lemma 2.3 and Lemma 2.2 (i) we infer that  $\lim_{t \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} q_n^n(t) \right) = 0$  and (ii) (d) is proven.  $\square$

## 5 Profile decomposition for bounded sequences in $W^{1,p}(\mathbf{R}^N)$

The proof of Theorem 1.7 relies on Theorem 1.5 and the following simple lemma.

**Lemma 5.1** *Assume that  $1 \leq p < \infty$  and  $q \in (p, p^*)$ , where  $p^* = \frac{Np}{N-p}$  if  $p < N$  and  $p^* = \infty$  if  $p \geq N$ . There exists  $C > 0$ , depending only on  $p, q$  and  $N$ , such that for any  $u \in W^{1,p}(\mathbf{R}^N)$*

there holds

$$\|u\|_{L^q(\mathbf{R}^N)} \leq C \left( \sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |\nabla u|^p + |u|^p dx \right)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{W^{1,p}(\mathbf{R}^N)}^{\frac{p}{q}}.$$

*Proof.* Lemma 5.1 is known by experts, we give the proof for the sake of completeness.

By the Sobolev embedding there is  $C_S > 0$  such that for any ball  $B \subset \mathbf{R}^N$  of radius 1 and any  $w \in W^{1,p}(B)$  there holds  $\|w\|_{L^q(B)} \leq C_S \|w\|_{W^{1,p}(B)}$ . Let  $u \in W^{1,p}(\mathbf{R}^N)$ . For any  $z \in \mathbf{R}^N$  we have

$$\begin{aligned} \int_{B(z,1)} |u|^q dx &\leq C_S^q \left( \int_{B(z,1)} |\nabla u|^p + |u|^p dx \right)^{\frac{q}{p}} \\ (5.1) \quad &\leq C_S^q \left( \sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |\nabla u|^p + |u|^p dx \right)^{\frac{q}{p}-1} \int_{B(z,1)} |\nabla u|^p + |u|^p dx. \end{aligned}$$

There is an integer  $K$  (depending only on  $N$ ) and there is a covering of  $\mathbf{R}^N$  by a family of balls  $(B(y_n, 1))_{n \in \mathbf{N}^*}$  such that each point belongs to at most  $K$  balls (that is,  $1 \leq \sum_{n=1}^{\infty} \mathbf{1}_{B(y_n, 1)} \leq K$ ). We write (5.1) for each ball  $B(y_n, 1)$ , then sum over  $n$  to get the desired conclusion.  $\square$

*Proof of Theorem 1.7.* We consider first the case  $1 < p < N$ . By the Sobolev embedding  $(u_n)_{n \geq 1}$  is bounded in  $L^{p^*}(\mathbf{R}^N)$ . Let  $\rho_n = |\nabla u_n|^p + |u_n|^p + |u_n|^{p^*}$ , so that  $(\rho_n)_{n \geq 1}$  is bounded in  $L^1(\mathbf{R}^N)$ . We use Theorem 1.5 for the sequence of measures  $(\mu_n)_{n \geq 1}$  with densities  $(\rho_n)_{n \geq 1}$  with respect to the Lebesgue measure.

If  $(\rho_n)_{n \geq 1}$  is a vanishing sequence it follows from Lemma 5.1 that  $u_n \rightarrow 0$  in  $L^q(\mathbf{R}^N)$  for any  $q \in (p, p^*)$  and the conclusion of Theorem 1.7 holds with  $V^i = 0$  for all  $i$ .

Assume that  $(\rho_n)_{n \geq 1}$  is not vanishing and consider the mapping  $j : \mathbf{N}^* \rightarrow \mathbf{N}^*$ , the sequences of balls  $(B(x_n^i, r_n^i))_{n \geq 1}$  and the numbers  $m_i$  given by Theorem 1.5. If there are only  $k$  positive  $m_i$ 's we put  $m_i = 0$  for  $i > k$ .

Let  $v_n^i = u_{j(n)}(\cdot + x_n^i)$ .

The sequence  $(v_n^1)_{n \geq 1}$  is bounded in  $W^{1,p}(\mathbf{R}^N)$ , hence there exists  $\kappa_1 : \mathbf{N}^* \rightarrow \mathbf{N}^*$  increasing and  $V^1 \in W^{1,p}(\mathbf{R}^N)$  such that

$$(5.2) \quad v_{\kappa_1(n)}^1 \rightharpoonup V^1 \quad \text{weakly in } W^{1,p}(\mathbf{R}^N) \text{ as } n \rightarrow \infty,$$

$$(5.3) \quad v_{\kappa_1(n)}^1 \rightarrow V^1 \quad \text{strongly in } L^q(B(0, R)) \text{ for any } q \in [1, p^*) \text{ and any } R > 0 \\ \text{and a.e. on } \mathbf{R}^N.$$

Fix  $R > 0$ . Since  $\nabla v_{\kappa_1(n)}^1 \rightharpoonup \nabla V^1$  weakly in  $L^p(B(0, R))$  and  $r_n^1 \rightarrow \infty$ , we get

$$\int_{B(0,R)} |\nabla V^1|^p dx \leq \liminf_{n \rightarrow \infty} \int_{B(0,R)} |\nabla v_{\kappa_1(n)}^1|^p dx \leq \liminf_{n \rightarrow \infty} \int_{B(x_{\kappa_1(n)}^1, r_{\kappa_1(n)}^1)} |\nabla u_{j(\kappa_1(n))}|^p dx.$$

Letting  $R \rightarrow \infty$  we find

$$(5.4) \quad \int_{\mathbf{R}^N} |\nabla V^1|^p dx \leq \liminf_{n \rightarrow \infty} \int_{B(x_n^1, r_n^1)} |\nabla u_{j(n)}|^p dx.$$



Using Fatou Lemma and proceeding similarly we discover

$$(5.5) \quad \int_{\mathbf{R}^N} |V^1|^q dx \leq \liminf_{n \rightarrow \infty} \int_{B(x_n^1, r_n^1)} |u_{j(n)}|^q dx \quad \text{for all } q \in [p, p^*].$$

Next we show that

$$(5.6) \quad \mathbb{1}_{B(0, r_{\kappa_1(n)}^1)} v_{\kappa_1(n)} \longrightarrow V^1 \quad \text{strongly in } L^q(\mathbf{R}^N) \text{ for any } q \in [p, p^*].$$

Fix  $\varepsilon > 0$ . Since  $\left(\mu_{j(n)|B(x_n^1, r_n^1)}\right)_{n \geq 1}$  concentrates around  $(x_n)_{n \geq 1}$ , there are  $r_\varepsilon > 0$  and  $n_\varepsilon \in \mathbf{N}$  such that  $\int_{B(x_n^1, r_n^1) \setminus B(x_n^1, r_\varepsilon)} \rho_{j(n)} dx < \varepsilon$  for all  $n \geq n_\varepsilon$ . This implies  $\int_{B(0, r_n^1) \setminus B(0, r_\varepsilon)} |v_n^1|^p + |v_n^1|^{p^*} dx < \varepsilon$  for  $n \geq n_\varepsilon$  and using Fatou Lemma we get  $\int_{\mathbf{R}^N \setminus B(0, r_\varepsilon)} |V^1|^p + |V^1|^{p^*} dx \leq \varepsilon$ . By interpolation we find  $\int_{\mathbf{R}^N \setminus B(0, r_\varepsilon)} |\mathbb{1}_{B(0, r_n^1)} v_n^1|^q dx < \varepsilon^{\nu_q}$  and  $\int_{\mathbf{R}^N \setminus B(0, r_\varepsilon)} |V^1|^q dx < \varepsilon^{\nu_q}$  for some  $\nu_q > 0$ . On the other hand, by (5.3) there is  $n'_\varepsilon \geq n_\varepsilon$  such that  $\|v_{\kappa_1(n)}^1 - V^1\|_{L^q(B(0, r_\varepsilon))} < \varepsilon$  for  $n \geq n'_\varepsilon$ . We conclude that  $\|\mathbb{1}_{B(0, r_n^1)} v_{\kappa_1(n)}^1 - V^1\|_{L^q(\mathbf{R}^N)} < \varepsilon + 2\varepsilon^{\frac{\nu_q}{q}}$  for all  $n \geq n'_\varepsilon$  and (5.6) is proven.

If  $m_2 = 0$  we take  $V^i = 0$  for  $i \geq 2$ . If  $m_2 > 0$ , proceeding as above we see that there exist an increasing mapping  $\kappa_2 : \mathbf{N}^* \rightarrow \mathbf{N}^*$  and  $V^2 \in W^{1,p}(\mathbf{R}^N)$  such that (5.2)–(5.6) hold with  $v_{\kappa_2(n)}^2$  and  $V^2$  instead of  $v_{\kappa_1(n)}^1$  and  $V^1$ , respectively. If  $m_\ell > 0$  for some  $\ell \in \mathbf{N}^*$ , by induction we find increasing mappings  $\kappa_1, \dots, \kappa_\ell : \mathbf{N}^* \rightarrow \mathbf{N}^*$  and  $V^1, \dots, V^\ell \in W^{1,p}(\mathbf{R}^N)$  such that

$$(5.7) \quad v_{\kappa_1(\dots(\kappa_\ell(n))\dots)}^\ell \rightharpoonup V^\ell \quad \text{weakly in } W^{1,p}(\mathbf{R}^N) \text{ as } n \rightarrow \infty,$$

$$(5.8) \quad v_{\kappa_1(\dots(\kappa_\ell(n))\dots)}^\ell \longrightarrow V^\ell \quad \text{strongly in } L^q(B(0, R)) \text{ for any } q \in [1, p^*) \text{ and } R > 0 \text{ and a.e. on } \mathbf{R}^N,$$

$$(5.9) \quad \int_{\mathbf{R}^N} |\nabla V^\ell|^p dx \leq \liminf_{n \rightarrow \infty} \int_{B(x_n^\ell, r_n^\ell)} |\nabla u_{j(n)}|^p dx,$$

$$(5.10) \quad \mathbb{1}_{B(0, r_{\kappa_1(\dots(\kappa_\ell(n))\dots)}^\ell)} v_{\kappa_1(\dots(\kappa_\ell(n))\dots)}^\ell \longrightarrow V^\ell \quad \text{strongly in } L^q(\mathbf{R}^N) \text{ for any } q \in [p, p^*].$$

If there is  $\ell_0$  such that  $m_{\ell_0} > 0$  and  $m_i = 0$  for  $i > \ell_0$ , we put  $V^i = 0$  for  $i > \ell_0$  and  $\kappa = \kappa_1 \circ \dots \circ \kappa_{\ell_0}$ . Otherwise we find  $V^i$  as above for all  $i \in \mathbf{N}^*$  and we put  $\kappa(n) = \kappa_1(\dots(\kappa_n(n))\dots)$ . We show that the conclusion of Theorem 1.7 holds true for the subsequence  $(u_{j(\kappa(n))})_{n \geq 1}$  and the functions  $V^i$  as above.

Fix  $\chi \in C_c^\infty(\mathbf{R}^N)$  such that  $0 \leq \chi \leq 1$ ,  $\chi = 1$  on  $B(0, \frac{1}{2})$  and  $\text{supp}(\chi) \subset B(0, 1)$ . For  $n \geq k$  the balls  $B(x_n^1, r_n^1), \dots, B(x_n^k, r_n^k)$  are disjoint and it is easy to see that

$$(5.11) \quad 1 = \chi\left(\frac{\cdot - x_n^1}{r_n^1}\right) + \dots + \chi\left(\frac{\cdot - x_n^k}{r_n^k}\right) + \prod_{i=1}^k \left(1 - \chi\left(\frac{\cdot - x_n^i}{r_n^i}\right)\right).$$

Fix  $k \in \mathbf{N}^*$  such that  $m_k > 0$ . For  $n \geq k$  we may write

$$\begin{aligned}
(5.12) \quad u_{j(\kappa(n))} &= \sum_{i=1}^k V^i(\cdot - x_{\kappa(n)}^i) + \sum_{i=1}^k \left( \chi \left( \frac{\cdot - x_{\kappa(n)}^i}{r_{\kappa(n)}^i} \right) u_{j(\kappa(n))} - V^i(\cdot - x_{\kappa(n)}^i) \right) \\
&\quad + \left( \prod_{i=1}^k \left( 1 - \chi \left( \frac{\cdot - x_{\kappa(n)}^i}{r_{\kappa(n)}^i} \right) \right) \right) u_{j(\kappa(n))}. \\
&= \sum_{i=1}^k V^i(\cdot - x_{\kappa(n)}^i) + v_{n,1}^k + v_{n,2}^k.
\end{aligned}$$

Let  $w_n^k = v_{n,1}^k + v_{n,2}^k$ . The fact that the sequence  $\left( \mathbb{1}_{B(x_{\kappa(n)}^i, r_{\kappa(n)}^i)} \rho_{j(\kappa(n))} \right)_{n \geq 1}$  concentrates around  $\left( x_{\kappa(n)}^i \right)_{n \geq 1}$  and (5.10) imply that  $\|v_{n,1}^k\|_{L^q(\mathbf{R}^N)} \rightarrow 0$  as  $n \rightarrow \infty$  for any  $q \in [p, p^*)$ . It is easy to see that

$$|v_{n,2}^k|^p + |\nabla v_{n,2}^k|^p \leq \rho_{j(\kappa(n))} \mathbb{1}_{\mathbf{R}^N \setminus \cup_{i=1}^k B(x_{j(\kappa(n))}^i, r_{j(\kappa(n))}^i)} + f_n^k,$$

where  $\|f_n^k\|_{L^1(\mathbf{R}^N)} \rightarrow 0$  as  $n \rightarrow \infty$ . Theorem 1.5 implies that

$$\limsup_{n \rightarrow \infty} \left( \sup_{z \in \mathbf{R}^N} \int_{B(z,1)} |v_{n,2}^k|^p + |\nabla v_{n,2}^k|^p dx \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Then using Lemma 5.1 we get  $\lim_{k \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} \|v_{n,2}^k\|_{L^q(\mathbf{R}^N)} \right) = 0$  for any  $q \in (p, p^*)$  and conclusion (ii) in Theorem 1.7 follows.

Using (5.10), the definition of  $v_{n,2}^k$  (see (5.12)) and the fact that  $\|v_{n,1}^k\|_{L^p(\mathbf{R}^N)} \rightarrow 0$  as  $n \rightarrow \infty$  we get

$$\begin{aligned}
\int_{\mathbf{R}^N} |u_{j(\kappa(n))}|^p dx &= \sum_{i=1}^k \int_{B(x_{\kappa(n)}^i, r_{\kappa(n)}^i)} |u_{j(\kappa(n))}|^p dx + \int_{\mathbf{R}^N \setminus \cup_{i=1}^k B(x_{\kappa(n)}^i, r_{\kappa(n)}^i)} |u_{j(\kappa(n))}|^p dx \\
&= \sum_{i=1}^k \|V^i\|_{L^p(\mathbf{R}^N)}^p + \|v_{n,2}^k\|_{L^p(\mathbf{R}^N)}^p + o(1) \\
&= \sum_{i=1}^k \|V^i\|_{L^p(\mathbf{R}^N)}^p + \|w_n^k\|_{L^p(\mathbf{R}^N)}^p + o(1) \quad \text{as } n \rightarrow \infty
\end{aligned}$$

and (iii) is proven.

Let  $h_n^i(x) = \chi \left( \frac{x}{r_{\kappa(n)}^i} \right) u_{j(\kappa(n))}(\cdot + x_{\kappa(n)}^i)$ . For all  $n$  such that  $r_{\kappa(n)}^i \geq 1$  we have

$$\begin{aligned}
(5.13) \quad &\int_{\mathbf{R}^N} \left| |\nabla u_{j(\kappa(n))}|^p - |\nabla v_{n,2}^k|^p - \sum_{i=1}^k |\nabla h_n^i(x - x_{\kappa(n)}^i)|^p \right| dx \\
&\leq C \sum_{i=1}^k \int_{B(x_{\kappa(n)}^i, r_{\kappa(n)}^i) \setminus B(x_{\kappa(n)}^i, \frac{1}{2} r_{\kappa(n)}^i)} |\nabla u_{j(\kappa(n))}|^p + |u_{j(\kappa(n))}|^p dx.
\end{aligned}$$

and the right hand side in the above inequality tends to zero as  $n \rightarrow \infty$  because the sequence  $\left( \mathbb{1}_{B(x_{\kappa(n)}^i, r_{\kappa(n)}^i)} \rho_{j(\kappa(n))} \right)_{n \geq 1}$  concentrates around  $(x_{\kappa(n)}^i)_{n \geq 1}$ .

Using (5.7) it is easy to see that  $h_n^i \rightharpoonup V^i$  weakly in  $W^{1,p}(\mathbf{R}^N)$ . Assume that  $p = 2$ . Then we have

$$\begin{aligned}
 (5.14) \quad & \int_{\mathbf{R}^N} |\nabla h_n^i(x - x_{\kappa(n)}^i)|^2 dx = \int_{\mathbf{R}^N} |\nabla h_n^i(x)|^2 dx \\
 & = \|\nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + \|\nabla h_n^i - \nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + 2\langle \nabla h_n^i - \nabla V^i, \nabla V^i \rangle_{L^2} \\
 & = \|\nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + \|\nabla h_n^i - \nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + o(1).
 \end{aligned}$$

Since  $r_{\kappa(n)}^i \rightarrow \infty$  for all  $i$  and  $|x_{\kappa(n)}^i - x_{\kappa(n)}^j| \rightarrow \infty$  as  $n \rightarrow \infty$ , it is straightforward to check that

$$(5.15) \quad \|\nabla v_{n,1}^k\|_{L^2(\mathbf{R}^N)}^2 = \sum_{i=1}^k \|\nabla h_n^i - \nabla V^i\|_{L^2(\mathbf{R}^N)}^2 + o(1) \quad \text{and}$$

$$(5.16) \quad \|\nabla w_n^k\|_{L^2(\mathbf{R}^N)}^2 = \|\nabla v_{n,1}^k\|_{L^2(\mathbf{R}^N)}^2 + \|\nabla v_{n,2}^k\|_{L^2(\mathbf{R}^N)}^2 + o(1)$$

as  $n \rightarrow \infty$ . Then (iv) follows from (5.13)–(5.16).

Assume next that  $N \leq p < \infty$ . Fix an increasing sequence  $(p_\ell)_{\ell \geq 1} \subset (p, \infty)$ ,  $p_\ell \rightarrow \infty$ . By the Sobolev embedding for any  $\ell$  there is  $C_\ell > 0$  such that for any  $u \in W^{1,p}(\mathbf{R}^N)$  there holds  $\|u\|_{L^{p_\ell}(\mathbf{R}^N)} \leq C_\ell \|u\|_{W^{1,p}(\mathbf{R}^N)}$ . Let  $A = \sup_{n \geq 1} \|u_n\|_{W^{1,p}(\mathbf{R}^N)}$ . Let

$$\rho_n = |\nabla u_n|^p + |u_n|^p + \sum_{\ell=1}^{\infty} \frac{1}{2^\ell C_\ell^{p_\ell} A^{p_\ell}} |u_n|^{p_\ell}.$$

Then  $(\rho_n)_{n \geq 1}$  is bounded in  $L^1(\mathbf{R}^N)$  and for any  $s \in [p, \infty)$  there exist  $a_s > 0$ ,  $\nu_s > 0$  such that for all  $n$  and all measurable sets  $E$ ,

$$\int_E |u_n|^s dx \leq a_s \left( \int_E \rho_n dx \right)^{\nu_s}.$$

We apply Theorem 1.5 to  $(\rho_n)_{n \geq 1}$ . The rest of the proof is as in the case  $p \in (1, N)$ . □

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